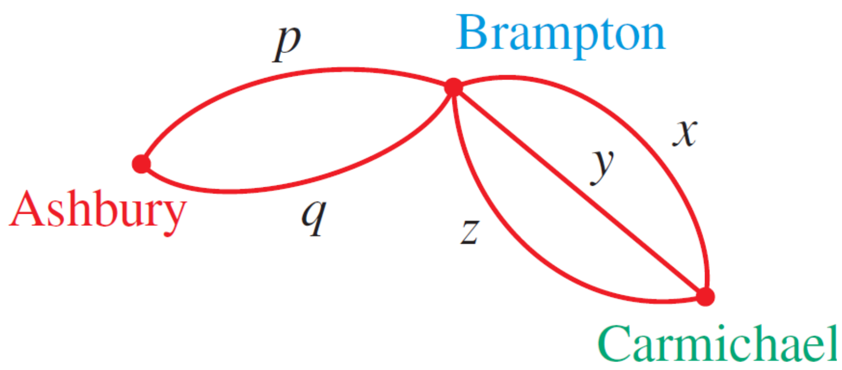
Objectives

- The Fundamental Counting Principle
- Counting Permutations
- Counting Combinations
- Problem Solving with Permutations and Combinations

The Fundamental Counting Principle

The Fundamental Counting Principle

Suppose that three towns—Ashbury, Brampton, and Carmichael—are located in such a way that two roads connect Ashbury to Brampton and three roads connect Brampton to Carmichael.



The Fundamental Counting Principle

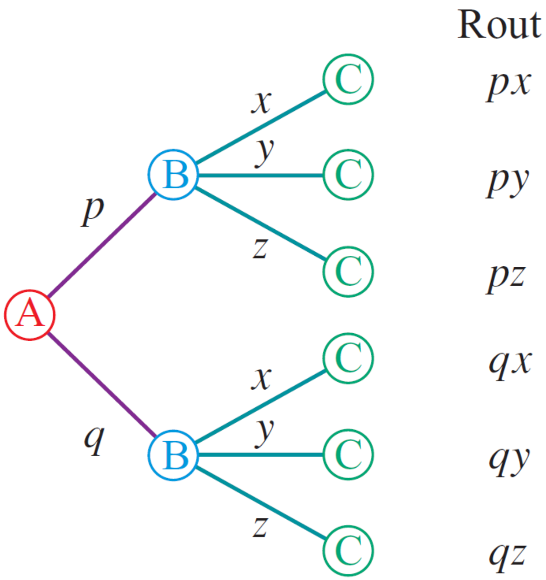
How many different routes can one take to travel from Ashbury to Carmichael via Brampton? The key to answering this question is to consider the problem in stages.

At the first stage—from Ashbury to Brampton—there are two choices. For each of these choices there are three choices at the second stage—from Brampton to Carmichael.

Thus the number of different routes is $2 \times 3 = 6$.

The Fundamental Counting Principle

These routes are conveniently enumerated by a *tree diagram* as in Figure 1.



Tree diagram

Figure 1

The Fundamental Counting Principle

The method that we used to solve this problem leads to the following principle.

THE FUNDAMENTAL COUNTING PRINCIPLE

Suppose that two events occur in order. If the first event can occur in m ways and the second can occur in n ways (after the first has occurred), then the two events can occur *in order* in $m \times n$ ways.

There is an immediate consequence of this principle for any number of events: If E_1, E_2,…., E_k are events that occur in order and if E_1 *can occur in* n_1 *ways*, E_2 *in* n_2 *ways*, and so on, then the events can occur in order in $n_1 \times n_2 \times \ldots . \times n_k$ ways.

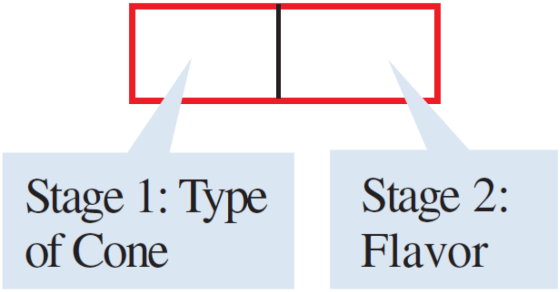
Example1 – *Using the Fundamental Counting Principle*

An ice-cream store offers three types of cones and 31 flavors. How many different single scoop ice-cream cones is it possible to buy at this store?

Solution:

There are two stages for selecting an ice-cream cone. At the first stage we choose a type of cone, and at the second stage we choose a flavor.

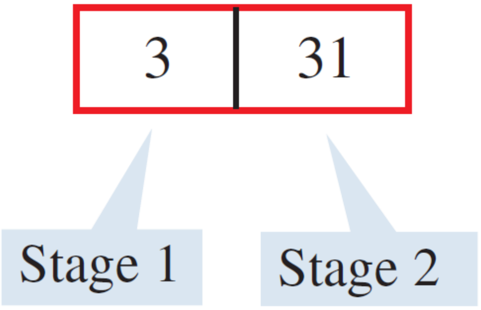
We can think of the different stages as boxes:



Example1 – *Solution*

cont'd

The first box can be filled in three ways, and the second can be filled in 31 ways:



By the Fundamental Counting Principle there are

$3 \times 31 = 93$ ways of choosing a single-scoop ice-cream cone at this store.

The Fundamental Counting Principle

Let S be a set with n elements. A subset of S can be chosen by making one of two choices for each element: We can choose the element to be *in* or *out* of A.

Since S has n elements and we have two choices for each element, by the Fundamental Counting Principle the total number of different subsets is $2 \times 2 \times \ldots \times 2$, where there are n factors.

This gives the following formula.

THE NUMBER OF SUBSETS OF A SET

A set with n elements has 2^n different subsets.

Example 3 – *Finding the Number of Subsets*

A pizza parlor offers a basic cheese pizza and a choice of 16 toppings. How many different kinds of pizza can be ordered at this pizza parlor?

Solution:

We need the number of possible subsets of the 16 toppings (including the empty set, which corresponds to a plain cheese pizza).

Thus

$$2^{16} = 65{,}536$$

different pizzas can be ordered.

Counting Permutations

Counting Permutations

A **permutation** of a set of distinct objects is an ordering of these objects. For example, some permutations of the letters *ABCD are*

ABDC *BACD* *DCBA* *DABC*

How many such permutations are possible?

There are four choices for the first position, three for the second (after the first has been chosen), two for the third (after the first two have been chosen), and only one choice for the fourth letter (the letter that has not yet been chosen).

Counting Permutations

By the Fundamental Counting Principle the number of possible permutations is

$$4 \times 3 \times 2 \times 1 = 4! = 24$$

The same reasoning with 4 replaced by *n* leads to the following.

The number of permutations of n objects is $n!$

How many permutations consisting of two letters can be made from these same four letters? Some of these permutations are *AB*, *AC*, *BD*, *DB*, ... There are 4 choices of the first letter and 3 for the second letter.

Counting Permutations

By the Fundamental Counting Principle there are 4 × 3 = 12 such permutations.

In general, if a set has n elements, then the number of ways of ordering r elements from the set is denoted by ***P*(*n*, *r*)** and is called **the number of permutations of *n* objects taken *r* at a time**.

PERMUTATIONS OF *n* OBJECTS TAKEN *r* AT A TIME

The number of permutations of n objects taken r at a time is

$$P(n, r) = \frac{n!}{(n - r)!}$$

Example 5 – *Finding the Number of Permutations*

A club has nine members. In how many ways can a president, vice president, and secretary be chosen from the members of this club?

Solution:

We need the number of ways of selecting three members, in order, for the positions of president, vice president, and secretary from the nine club members.

This number is

$$P(9, 3) = \frac{9!}{(9-3)!} = \frac{9 \times 8 \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}$$

$$= 504$$

Counting Combinations

Counting Combinations

A **combination** of *r* elements of a set is any subset of *r* elements from the set (without regard to order).

If the set has *n* elements, then the number of combinations of *r* elements is denoted by ***C*(*n*, *r*)** and is called the **number of combinations of n elements taken *r* at a time.**

For example, consider a set with the four elements *A*, *B*, *C*, and *D*. The combinations of these four elements taken three at a time are listed below.

ABC *ABD* *ACD* *BCD*

Counting Combinations

We notice that the number of combinations is a lot fewer than the number of permutations. In fact, each combination of three elements generates 3! permutations.

So, $C(4, 3) = P(4, 3)/3! = 4$.

In general, each combination of r objects gives rise to r! permutations of these objects, so we get the following formula.

COMBINATIONS OF n OBJECTS TAKEN r AT A TIME

The number of combinations of n objects taken r at a time is

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

Counting Combinations

The key difference between permutations and combinations is order.

If we are interested in ordered arrangements, then we are counting permutations, but if we are concerned with subsets without regard to order, then we are counting combinations.

Compare Example 7 (where order doesn’t matter) to Example 5 (where order does matter).

Example 7 – *Finding the Number of Combinations*

A club has nine members. In how many ways can a committee of three be chosen from the members of this club?

Solution:

We need the number of ways of choosing three of the nine members. Order is not important here, because the committee is the same no matter how its members are ordered.

So we want the number of combinations of nine objects (the club members) taken three at a time.

Example 7 – *Solution*

cont'd

This number is

$$C(9, 3) = \frac{9!}{3!(9-3)!}$$

$$= \frac{9 \times 8 \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{(3 \times 2 \times 1) \times (\cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1})}$$

$$= 84$$

Problem Solving with Permutations and Combinations

Problem Solving with Permutations and Combinations

The crucial step in solving counting problems is deciding whether to use permutations, combinations, or the Fundamental Counting Principle.

In some cases the solution of a problem may require using more than one of these principles.

Here are some general guidelines to help us decide how to apply these principles.

GUIDELINES FOR SOLVING COUNTING PROBLEMS

1. **Fundamental Counting Principle.** When consecutive choices are being made, use the Fundamental Counting Principle.
2. **Does Order Matter?** When we want to find the number of ways of picking r objects from n objects, we need to ask ourselves, "Does the order in which we pick the objects matter?"

   If the order matters, we use permutations.

   If the order doesn't matter, we use combinations.

Example 10 – *Using Permutations and Combinations*

A committee of seven—consisting of a chairman, a vice chairman, a secretary, and four other members—is to be chosen from a class of 20 students. In how many ways can the committee be chosen?

Solution:

In choosing the three officers, order is important. So the number of ways of choosing them is

$$P(20, 3) = 6840$$

Next, we need to choose four other students from the 17 remaining.

Example 10 – *Solution*

cont'd

Since order doesn't matter in choosing these four members, the number of ways of doing this is

$$C(17, 4) = 2380$$

By the Fundamental Counting Principle the number of ways of choosing this committee is

$$P(20, 3) \times C(17, 4) = 6840 \times 2380$$

$$= 16{,}279{,}200$$